

# Thermodynamics of nonlinear charged Lifshitz black branes with hyperscaling violation

M. H. Dehghani <sup>1,2</sup>, A. Sheykhi <sup>1,2</sup>, and S. E. Sadati <sup>1</sup>

<sup>1</sup> *Physics Department and Biruni Observatory,*

*College of Sciences, Shiraz University, Shiraz 71454, Iran*

<sup>2</sup> *Research Institute for Astronomy and Astrophysics of Maragha (RIAAM),*

*P.O. Box 55134-441, Maragha, Iran*

## Abstract

In this paper, we investigate the thermodynamics of hyperscaling violating Lifshitz black branes in the presence of a nonlinear massless electromagnetic field. We, first, obtain analytic nonlinear charged black brane solutions with hyperscaling violating factor in dilaton gravity and give the condition on the parameters of the metric for having black brane solutions. Second, we introduce the appropriate finite action in grand-canonical and canonical ensembles for nonlinear electromagnetic field. Next, by generalizing the counterterm method for the asymptotic Lifshitz spacetimes with hyperscaling violating factor, we calculate the energy density of our solutions. Then, we present a relation between the energy density and the thermodynamic quantities, electric potential, charge density, temperature and entropy density. This relation is the generalization of Smarr formula for anti-de Sitter black branes and charged Lifshitz solutions. Finally, we perform a stability analysis in both the canonical and grand-canonical ensemble. We show that the nonlinearity of electromagnetic field can make the solutions unstable in grand-canonical ensemble.

## I. INTRODUCTION

Gauge/gravity duality [1] may be thought of as a practical tool to study strongly coupled systems near critical points where the system exhibits a scaling symmetry. Generally, at a critical point the system may be described by a conformal field theory (CFT). From the gauge/gravity point of view this means that the gravitational theory is defined on a metric which is asymptotically locally anti-de Sitter (AdS). On the other hand, the central idea in gauge/gravity duality is that each state in the bulk has a corresponding state in the dual field theory. In particular, black objects are dual to thermal ensembles in the field theory with the same thermodynamic properties (temperature, entropy, chemical potential, etc.) as the bulk spacetime. Most notably, the holographic techniques are also useful to study condensed matter systems at strong coupling [2].

Although the idea of holography has been first used in the AdS/CFT correspondence, largely inspired by condensed matter systems these techniques have been brought to bear on other types of systems too. As a first generalization, one may consider metrics which can be dual to scale-invariant field theories which are, however, not conformally invariant, but instead enjoy a dynamical critical exponent  $z \neq 1$  [3]:

$$ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}_{d-2}^2. \quad (1)$$

This metric which is known as Lifshitz metric is invariant under the following scaling

$$t \rightarrow \lambda^z t, \quad r \rightarrow \lambda^{-1} r \quad x^i \rightarrow \lambda x^i. \quad (2)$$

For  $z = 1$ , the scaling is isotropic; it corresponds to relativistic invariance. In [3], it was proposed that gravity duals of field theories with Lifshitz scaling should have metric solutions that asymptote to Lifshitz metric give in Eq. (1). Indeed, the Lifshitz black holes have been found to emerge as gravity duals of some condensed matter systems with anisotropic scaling symmetry (2). It is well known that the Einstein-Hilbert (EH) action does not admit Lifshitz geometry with or without hyperscaling violation. In order to have Lifshitz solutions, one may add a massive gauge field to the EH action [4–7]. Another way of constructing asymptotically Lifshitz solutions is through the use of a dilaton field [8]. The exact asymptotically Lifshitz charged solutions of Einstein-dilaton-Maxwell gravity have been introduced in [9].

Another generalization which has been recently got more attention is the study of systems with hyperscaling violation. Indeed, by including both dilaton and Abelian gauge fields, it

is possible to find even more interesting metrics, which on the top of an anisotropic scaling, have also an overall hyperscaling violating factor. More precisely, one may have a geometry in the form of [10]

$$ds^2 = r^{\frac{-2\theta}{d}} \left( -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}_{d-2}^2 \right), \quad (3)$$

where the constants  $z$  and  $\theta$  are dynamical and hyperscaling violating exponents, respectively. This is the most general geometry which is spatially homogeneous and covariant under the scale-transformation (2) and is not scale invariant, but transforms as

$$ds \rightarrow \lambda^{\frac{\theta}{d}} ds, \quad (4)$$

under the transformations (2). The concept of hyperscaling violation has also developed in condensed matter and in the context of gauge gravity duality [11]. For example, hyperscaling violation provides a useful way to realize compressible matters in  $(2+1)$ -dimensions and to pursue the holographic realization of systems with Fermi surfaces [12]. The EH action with Abelian gauge field coupled to a dilaton may also have black hole solutions which asymptote to the hyperscaling violating Lifshitz metric given in Eq. (3). For asymptotically Lifshitz spacetimes with hyperscaling violation, black hole solutions were obtained in [13].

Although an exact asymptotically Lifshitz black brane solution with hyperscaling violating factor has been presented in [13], its thermodynamics has not been investigated. In the present paper we would like to investigate the effects of adding a power-law Maxwell invariant term to the action of Einstein-Maxwell in the presence of a massless scalar dilaton field. This higher term,  $(-F_{\mu\nu}F^{\mu\nu})^s$ , also appears in the low-energy limit of heterotic string theory [14]. Also, it is worth noting that this term for  $s = d/4$ , is conformally invariant [14]. We also like to investigate the thermodynamics of hyperscaling violating Lifshitz black branes in the presence of a nonlinear massless gauge field. In order to do this, we should generalize the counterterm method for Lifshitz black holes in the presence of a massive electromagnetic field introduced in [15]. We introduce the appropriate action of this theory in both the canonical and grand-canonical ensembles. The appropriate action of Einstein gravity both in canonical and grand-canonical ensembles in the presence of a linear gauge field and in the absence of dilaton has been introduced by Hawking and Ross [16]. This action has been generalized in [17] for nonlinear electromagnetic field. Our first aim, here, is to generalize this action for linear electromagnetic field to the case of nonlinear gauge field both in canonical and grand-canonical ensembles in the presence of dilaton. Having the

appropriate action in hand, we use the counterterm method to calculate the finite energy density of the black brane solutions.

This paper is outlined as follows. In the next section we present the action and obtain the basic field equations by varying the action. In section III, we study charged black brane solutions with hyperscaling violation in dilaton gravity in the presence of nonlinear electromagnetic field. In Sec. IV, we study the finite action of the theory in both the canonical and grand-canonical ensemble. The thermodynamics of charged hyperscaling violating black branes will be investigated in Sec. V. We also generalize the Smarr formula of black branes to the case of asymptotically Lifshitz charged black branes with hyperscaling violation. In Sec. VI, we perform a thermal stability analysis of the solutions. We finish our paper with some concluding remarks in section VI.

## II. FIELD EQUATION

In this section we review the field equations of Einstein-dilaton gravity in the presence of a linear and a nonlinear electromagnetic field. The gravity part of the action in  $(d+2)$ -dimensions may be written as

$$I = \frac{1}{16\pi} \int_M d^{d+2}x \sqrt{-g} R + \frac{1}{8\pi} \int_{\partial M} d^{d+1}x \sqrt{-h} K + I_M, \quad (5)$$

where  $R$  is Ricci scalar,  $\partial M$  is the hypersurface at some constant  $r$ ,  $h_{\alpha\beta}$  is the induced metric,  $K$  is the trace of the extrinsic curvature  $K_{\mu\nu} = \nabla_{(\mu} n_{\nu)}$  of the boundary and the unit vector  $n^\mu$  is orthogonal to the boundary and outward directed. The second term is the well-known Gibbons-Hawking term, which is added to the action in order to have a well-defined variational principle. The matter part of the action is

$$I_M = \frac{1}{16\pi} \int_M d^{d+2}x \sqrt{-g} \left[ -\frac{1}{2}(\partial\Phi)^2 + V(\Phi) - \frac{1}{4}e^{\lambda_1\Phi} H_{\mu\nu} H^{\mu\nu} + \frac{1}{4}e^{\lambda_2\Phi} (-F)^s \right], \quad (6)$$

where  $\lambda_1$  and  $\lambda_2$  are free parameters of the model,  $\Phi$  is the dilaton field,  $F = F_{\mu\nu} F^{\mu\nu}$  is the Maxwell invariant,  $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$  and  $A_\mu$  is the nonlinear electromagnetic field. The linear electromagnetic field  $H_{\mu\nu} = \partial_{[\mu} B_{\nu]}$  with the Abelian gauge field  $B_\nu$  together with the dilaton field are to make the asymptotic of the geometry to be the desired one. The nonlinear electromagnetic field is required to have a nonlinear charged solution. For the potential of the scalar field, motivated by the typical exponential potentials of string theory,

we will consider the following potential

$$V(\Phi) = -2\Lambda e^{\gamma\Phi}, \quad (7)$$

where  $\Lambda$  and  $\gamma$  are two free parameters. The variation of the total action  $I = I_G + I_M$  with respect to gravitational field  $g^{\mu\nu}$ , the scalar field  $\Phi$  and the gauge fields  $A_\mu$  and  $B_\mu$  yields

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{V(\Phi)}{d}g_{\mu\nu} + \frac{1}{2}e^{\lambda_1\Phi}\left(H^\rho{}_\mu H^\rho{}_{\nu} - \frac{g_{\mu\nu}}{2d}H_{\lambda\rho}H^{\lambda\rho}\right) + \frac{1}{2}e^{\lambda_2\Phi}\left[s(-F)^{s-1}F^\rho{}_\mu F^\rho{}_{\nu} + \left(\frac{2s-1}{2d}\right)(-F)^s g_{\mu\nu}\right] \quad (8)$$

$$\nabla^2\Phi = -\frac{dV(\Phi)}{d\Phi} + \frac{1}{4}\lambda_1 e^{\lambda_1\Phi}H_{\mu\nu}H^{\mu\nu} - \frac{1}{4}\lambda_2 e^{\lambda_2\Phi}(-F)^s, \quad (9)$$

$$\partial_\mu(\sqrt{-g}e^{\lambda_1\Phi}H^{\mu\nu}) = 0, \quad (10)$$

$$\partial_\mu(s\sqrt{-g}e^{\lambda_2\Phi}(-F)^{s-1}F^{\mu\nu}) = 0. \quad (11)$$

### III. BLACK BRANE SOLUTIONS WITH HYPERSCALING VIOLATION

To find the nonlinear charged hyperscaling violating Lifshitz black branes, we will closely follow the approach of [13]. A suitable ansatz for the metric, the dilaton field and the gauge fields of an isotropic static spacetime may be written as

$$ds^2 = r^{2\alpha}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2\sum_{i=1}^d dx_i^2\right), \quad (12)$$

$$\Phi = \Phi(r), \quad F_{rt} = g(r), \quad H_{rt} = h(r), \quad (13)$$

while the other components of gauge fields are set to be zero. From the Maxwell Eqs. (10) and (11), using the above ansatz, one obtains

$$H_{rt} = q_1 e^{-\lambda_1\Phi} r^{\alpha(2-d)+z-d-1}, \quad (14)$$

$$F_{rt} = q_2 e^{\frac{-\lambda_2\Phi}{2s-1}} r^{\frac{(2s-1)(z+2\alpha-1)-d\alpha-d}{2s-1}}. \quad (15)$$

Combining the  $tt$  and  $rr$  components of Eq. (8), one obtains

$$r^2(\partial_r\Phi)^2 = 2d(1+\alpha)(z+\alpha-1) \equiv \beta^2. \quad (16)$$

Thus, the scalar field is

$$\Phi(r) = \ln\left(\frac{r}{r_0}\right)^\beta. \quad (17)$$

Note that in order to have a real dilaton field, one has to assume  $(1 + \alpha)(z + \alpha - 1) \geq 0$ . Indeed, it can be seen that this assumption is also the consequence of the null energy condition. More precisely consider a null vector as  $\xi^\mu = (\sqrt{g^{rr}}, \sqrt{g^{tt}}, \mathbf{0})$ , then

$$T_{\mu\nu}\xi^\mu\xi^\nu \propto d(1 + \alpha)(z + \alpha - 1)r^{-2\alpha}f(r), \quad (18)$$

which is positive provided  $(1 + \alpha)(z + \alpha - 1) \geq 0$ . To find the metric function  $f(r)$  one may use any of the components of the field equations (8). Using the  $(ii)$ -component of Eq. (8) and replacing  $\Phi(r)$  from Eq. (17), we arrive at

$$\begin{aligned} [r^{d(\alpha+1)+z}f(r)]' &= \frac{r^{\alpha(d+2)+z+d-1}}{(\alpha+1)d} \left\{ -2\Lambda r_0^{-\gamma\beta} r^{\gamma\beta} \right. \\ &\quad \left. - \frac{1}{2}r_0^{\lambda_1\beta} q_1^2 r^{-2d(\alpha+1)-\lambda_1\beta} - \frac{1}{4}(2s-1)(2q_2^2)^s r_0^{\frac{\lambda_2\beta}{2s-1}} r^{\frac{-\lambda_2\beta-2sd(\alpha+1)}{2s-1}} \right\}. \end{aligned} \quad (19)$$

The above equation can be integrated to find the function  $f(r)$  as

$$\begin{aligned} f(r) &= -\frac{m}{r^{d+z+\alpha d}} + \frac{-2\Lambda r_0^{-\beta\gamma} r^{\beta\gamma+2\alpha}}{d(\alpha+1)(\gamma\beta + \alpha(d+2) + z + d)} \\ &\quad - q_1^2 r_0^{\lambda_1\beta} \frac{r^{-2\alpha(d-1)-\beta\lambda_1-2d}}{2d(\alpha+1)[\alpha(2-d) + z - d - \beta\lambda_1]} \\ &\quad - \frac{1}{4d} \frac{(2q_2^2)^s (2s-1)^2 r_0^{\frac{\lambda_2\beta}{2s-1}} r^{\frac{(-2+(4-2d)s)\alpha-2sd-\lambda_2\beta}{2s-1}}}{[(4s-d-2)\alpha - d - z + 2sz - \lambda_2\beta](\alpha+1)}. \end{aligned}$$

where  $m$  is the integration constant which is related to the mass of the black brane as we will see later. In order to have the desired asymptotic behavior for the metric,  $f(r)$  should be equal to 1 at infinity. Thus, we should have

$$\begin{aligned} \gamma &= -\frac{2\alpha}{\beta}, \\ \Lambda &= -\frac{1}{2}r_0^{-2\alpha}(d(\alpha+1) + z - 1)(d(\alpha+1) + z). \end{aligned}$$

The function  $f(r)$  should satisfy the equation of motion of the scalar field. That is

$$\begin{aligned} 8\Lambda(\gamma d(\alpha+1) + \beta)r^{\gamma\beta}r_0^{-\gamma\beta} + 2(-\lambda_1 d(\alpha+1) + \beta)r_0^{\beta\lambda_1}r^{-\beta\lambda_1-2d(\alpha+1)}q_1^2 \\ + (-\lambda_2 d(\alpha+1) + \beta(2s-1))(2q_2^2)^s r_0^{\frac{\beta\lambda_2}{2s-1}} r^{\frac{-\beta\lambda_2-2sd(\alpha+1)}{2s-1}} = 0. \end{aligned} \quad (20)$$

This equation can be solved for the parameters  $\lambda_1, \lambda_2$  and  $q_1$ . One may find

$$\begin{aligned}\lambda_1 &= -\frac{2\alpha(d-1)+2d}{\sqrt{2d(\alpha+1)(\alpha+z-1)}}, \\ \lambda_2 &= (2s-1)\sqrt{\frac{2(\alpha+z-1)}{d(\alpha+1)}}, \\ q_1^2 &= \frac{-4\Lambda(z-1)r_0^{2d(\alpha+1)}}{d(\alpha+1)+z-1}.\end{aligned}\tag{21}$$

Indeed, we have fixed the parameters  $\Lambda, \gamma, \lambda_1, \lambda_2$  and  $q_1$ , while  $q_2$  remains as an undetermined free parameter which is related to the charge of the solution. It is, now, important to check whether the other equations hold without imposing any further constraints on the parameters of the solution. In particular one of the nontrivial equations that need to be checked is the  $tt$  component of the Einstein equations of motion. Indeed, it is easy to see that this equation is also satisfied without imposing any further constraints.

To summarize, the field equations admit the following nonlinear charged black brane solution with hyperscaling violating factor. Note that in order to follow the standard notation in the literature we set  $\alpha = -\theta/d$ , where  $\theta$  is the hyperscaling violating exponent. It is also notable to mention that these solutions are not valid for  $\theta = d$  where  $\alpha = -1$ . Finally, we can rewrite the function  $f(r)$  as

$$f(r) = 1 - \frac{m}{r^{z+d-\theta}} + \frac{q^{2s}}{r^{\Gamma+d+z-\theta}},\tag{22}$$

where

$$q^{2s} = \frac{(2s-1)r_0^{2(z-1-\frac{\theta}{d})}}{4(d-\theta)\Gamma}(2q_2^2)^s,\tag{23}$$

$$\Gamma = z-2 + \frac{d-\theta}{2s-1}.\tag{24}$$

It is worth mentioning that since  $f(r)$  should go to 1 at infinity, we should have  $z+d-\theta > 0$  and  $\Gamma+d+z-\theta > 0$ . But, as we will show later  $\Gamma > 0$  and therefore  $f(r) \rightarrow 1$  as  $r$  goes to infinity provided  $z+d-\theta > 0$ . The gauge and dilaton fields are now given by

$$H_{rt} = q_1 r_0^{2(-\frac{\theta}{d}+\theta-d)} r^{(d-\theta+z-1)},\tag{25}$$

$$F_{rt} = q_2 r_0^{2(-\frac{\theta}{d}+z-1)} r^{-(\Gamma+1)},\tag{26}$$

$$\Phi = \ln\left(\frac{r}{r_0}\right)^{\sqrt{2(d-\theta)(-\frac{\theta}{d}+z-1)}}.\tag{27}$$

$$q_1^2 = 2(z-1)(z+d-\theta)r_0^{2(\frac{\theta}{d}+d-\theta)}. \quad (28)$$

Equation (28) shows that since  $z+d-\theta > 0$ ,  $z$  should be larger than 1 too.

#### IV. FINITE ACTION IN CANONICAL AND GRAND-CANONICAL ENSEMBLES

In general, the total action  $I$  given in Eq. (5) is divergent when evaluated on a solution. One way of dealing with the divergences of the action is adding some counterterms to the action (5). The counterterms should contain a part which removes the divergence of the gravity part of the action and a part for dealing with the divergence of the matter action. Since the horizon of our solution is flat, the counterterm which removes the divergence of the gravity part should be proportional to  $r^{\theta/d}\sqrt{h}$ . The counterterm for the matter part of the action in the absence of dilaton ( $\lambda_1 = 0$ ) and for the case of Lifshitz solution ( $\theta = 0$ ) has been introduced in Ref. [15]. Here, we generalize it to the case of Lifshitz solutions with hyperscaling violating factor in the presence of dilaton field. For this case, due to the fact that on the boundary  $e^{\lambda_1\Phi(r)}B_\gamma B^\gamma$  is constant for our solutions, the following counterterms make the action finite:

$$I_{ct} = -\frac{1}{16\pi} \int_{\partial M} d^{d+1}x \sqrt{-h} r^{\theta/d} \left[ 2(d-\theta) - a \left( -e^{\lambda_1\Phi} B_\gamma B^\gamma \right)^{1/2} \right],$$

where  $a = \sqrt{2(z-1)(d+z-\theta)}$ . Note that because of the constraints on  $d$ ,  $z$  and  $\theta$  explained in the last section,  $a$  is real as it should be. The variation of the total action ( $I_{tot} = I + I_{ct}$ ) about the solutions of the equations of motion is

$$\delta I_{tot} = \int d^{d+1}x \left( S_{\alpha\beta} \delta h^{\alpha\beta} + S_\alpha^L \delta B^\alpha \right) + \frac{1}{16\pi} \int d^{d+1}x \sqrt{-h} s (-F)^{s-1} e^{\lambda_2\Phi} n^\mu F_{\mu\alpha} \delta A^\alpha, \quad (29)$$

where

$$S_{\alpha\beta} = \frac{\sqrt{-h}}{16\pi} \left\{ \Pi_{\alpha\beta} + r^{\theta/d} \left[ (d-\theta) h_{\alpha\beta} - \frac{a}{2} e^{\lambda_1\Phi/2} (-B_\gamma B^\gamma)^{-1/2} (B_\alpha B_\beta - B_\gamma B^\gamma h_{\alpha\beta}) \right] \right\}, \quad (30)$$

$$S_\beta^L = -\frac{\sqrt{-h}}{16\pi} \left\{ n^\alpha H_{\alpha\beta} + a e^{\lambda_1\Phi/2} (-B_\gamma B^\gamma)^{-1/2} B_\beta \right\}, \quad (31)$$

with  $\Pi_{\alpha\beta} = K_{\alpha\beta} - K h_{\alpha\beta}$ .

Equation (29) shows that the variation of the total action with respect to  $A^\mu$  will only give the equation of motion of the nonlinear massless field  $A^\mu$  provided the variation is at fixed



nonlinear massless gauge potential on the boundary. Thus, the total action,  $I_{tot} = I + I_{ct}$ , given in Eqs. 29 is appropriate for the grand-canonical ensemble, where  $\delta A^\mu = 0$  on the boundary. But in the canonical ensemble, where the electric charge  $s(-F)^{s-1}e^{\lambda_2\Phi}n^\mu F_{\mu\alpha}A^\alpha$  is fixed on the boundary, the appropriate action is

$$I_{tot} = I + I_{ct} - \frac{1}{16\pi} \int_{\partial M} d^{d+1}x \sqrt{-h} s e^{\lambda_2\Phi} n^\mu (-F)^{s-1} F_{\mu\alpha} A^\alpha. \quad (32)$$

The last term in Eq. (32) is the generalization of the boundary term introduced by Hawking for linear electromagnetic field [16] and in Ref. [17] for nonlinear gauge field. Thus, both in canonical and grand-canonical ensemble, the variation of total action about the solutions of the field equations is

$$\delta I_{tot} = \int d^{d+1}x (S_{\alpha\beta} \delta h_{\alpha\beta} + S_\alpha^L \delta B^\alpha). \quad (33)$$

That is, the nonlinear gauge field is absent in the variation of the total action both in canonical and grand-canonical ensembles, and therefore, as in the absence of massless electromagnetic field, the dual field theory for a hyperscaling violating Lifshitz spacetime in the presence of a nonlinear electromagnetic field has a stress tensor complex consisting of the energy density  $\varepsilon$ , energy flux  $\varepsilon_i$ , momentum density  $P_i$  and spatial stress tensor  $P_{ij}$  satisfying the conservation equations

$$\partial_t \varepsilon + \partial_i \varepsilon^i = 0, \quad \partial_t P_j + \partial_i P_j^i = 0, \quad (34)$$

$$\varepsilon = 2S_t^t - S_L^t B_t, \quad \varepsilon^i = 2S_t^i - S_L^i B_t, \quad (35)$$

$$P_i = -2S_i^t + S_L^t B_i, \quad P_i^j = -2S_i^j + S_L^j B_i. \quad (36)$$

## V. THERMODYNAMICS OF NONLINEAR CHARGED HYPERSCALING VIOLATING BLACK BRANES

Now, we investigate thermodynamics of charged Lifshitz black branes with hyperscaling violating factor. One can obtain the temperature of the event horizon by using

$$T = \frac{1}{2\pi} \left( -\frac{1}{2} \nabla_b \zeta_a \nabla^b \zeta^a \right)_{r=r_+}^{1/2}. \quad (37)$$

Using the fact that the mass parameter is

$$m = r_+^{z-\theta+d} + q^{2s} r_+^{-\Gamma}, \quad (38)$$

one obtains

$$T = \frac{1}{4\pi} \left\{ (d + z - \theta) r_+^z - \Gamma q^{2s} r_+^{-(\Gamma+d-\theta)} \right\}. \quad (39)$$

One may note that there exists an extreme black hole. The charge and mass of the extreme black hole is

$$\begin{aligned} q_{ext}^{2s} &= \frac{z + d - \theta}{\Gamma} r_{ext}^{z+d-\theta+\Gamma}, \\ m_{ext} &= \frac{\Gamma + z + d - \theta}{\Gamma} r_{ext}^{z+d-\theta}, \end{aligned}$$

and therefore the condition of having black holes is

$$\left( \frac{\Gamma m}{\Gamma + z + d - \theta} \right)^{\Gamma+z+d-\theta} \geq \left( \frac{\Gamma q^{2s}}{z + d - \theta} \right)^{z+d-\theta}. \quad (40)$$

The entropy per unit volume of the horizon is given, as usual, by the Bekenstein-Hawking formula

$$S = \frac{1}{4} r_+^{d-\theta}. \quad (41)$$

Using Eq. (35) one obtains the energy density of black brane as

$$\varepsilon = \frac{d - \theta}{16\pi} m, \quad (42)$$

where  $m$  in terms of  $r_+$  and  $q$  is given in Eq. (38).

Before starting the calculations of the electric charge and potential, one should note that the linear gauge field  $B_\mu$  is just needed to support the structure of the asymptotic of the Lifshitz spacetime solution with hyperscaling violating factor. In other words, it does not have a thermodynamic interpretation. The fact that this gauge field does not affect the thermodynamics is due to the fact that its charge parameter is not a free parameter and it is completely determined by the parameters of the metric. The electric charge density of the gauge field  $A_\mu$  may be calculated by using

$$Q = \frac{1}{16\pi\Omega} \int d\Omega r^{d-\theta} e^{\lambda_2 \Phi} (-F)^{s-1} F_{\mu\nu} n^\mu u^\nu,$$

where  $n^\mu$  and  $u^\nu$  are the spacelike and timelike unit normals to a sphere of radius  $r$ ,

$$u^\nu = \frac{1}{\sqrt{-g_{tt}}} dt = \frac{1}{r^z \sqrt{f}} dt, \quad n^\mu = \frac{1}{\sqrt{g_{rr}}} dr = r \sqrt{f} dr.$$

One obtains

$$Q = \frac{s}{16\pi} 2^{s-1} (q_2)^{2s-1}. \quad (43)$$

Choosing the infinity as the reference point of the potential, the nonlinear gauge field  $A_t$  can be obtained as

$$A_t = \int_{\infty}^r F_{rt} dr = -\frac{1}{\Gamma} r_0^{2(-\frac{\theta}{d}+z-1)} \frac{q_2}{r^\Gamma},$$

where  $\Gamma$  is given in Eq. (24). To avoid the divergence of  $A_t$ , we must have  $\Gamma > 0$ . Now, using the definition of electric potential at infinity with respect to the horizon

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}, \quad (44)$$

where  $\chi^\mu = \partial/\partial t$  is the null generators of the event horizon, the electric potential is obtained as

$$U = \frac{q_2}{\Gamma} r_0^{2(-\frac{\theta}{d}+z-1)} r_+^{-\Gamma}. \quad (45)$$

Now, it is a matter of straightforward calculation to show that the first law of thermodynamics holds on the black brane horizon:

$$d\varepsilon = TdS + UdQ. \quad (46)$$

Specifically, the temperature and potential given in Eqs. (39) and (45) can be reobtained by the following equations:

$$T = \left( \frac{\partial \varepsilon}{\partial S} \right)_Q, \quad U = \left( \frac{\partial \varepsilon}{\partial Q} \right)_S. \quad (47)$$

The Smarr formula can be obtained as

$$\varepsilon = \frac{(d-\theta)}{(d-\theta+z)} \left[ TS + \frac{(\Gamma+z+d-\theta)(2s-1)}{2s(d-\theta)} Q\Phi \right]. \quad (48)$$

## VI. STABILITY OF NONLINEAR CHARGED BLACK BRANES WITH HYPER-SCALING VIOLATION

Finally, we investigate the stability of charged black brane solutions of Einstein nonlinear Maxwell gravity both in canonical and grand-canonical ensemble. The local stability can in principle be carried out by finding the determinant of the Hessian matrix of  $\varepsilon(S, Q)$  with respect to its extensive variables  $S$  and  $Q$  ( $H_{X_i X_j}^\varepsilon = [\partial^2 \varepsilon / \partial X_i \partial X_j]$ ) [18]. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter, and therefore the positivity of the heat capacity  $C_Q =$

$T_+/(\partial^2 \varepsilon / \partial S^2)_Q$  is sufficient to ensure the local stability. The heat capacity can be obtained as

$$\begin{aligned} C_Q &= T \left( \frac{\partial S}{\partial T} \right)_Q = \left( \frac{\partial \varepsilon / \partial r_+}{\partial T / \partial r_+} \right)_Q \\ &= \frac{(d - \theta) T r_+^{d-\theta}}{z(z + d - \theta) r_+^z + \Gamma [\Gamma + d - \theta] q^{2s} r_+^{-(d-\theta+\Gamma)}}. \end{aligned} \quad (49)$$

Since as we mentioned before both  $z + d - \theta$  and  $\Gamma$  are positive, we conclude that the specific heat at constant charge is always positive provided  $d > \theta$ . That is, the black brane solutions are stable in canonical ensemble for  $d > \theta$ .

In the grand-canonical ensemble, the determinant of the Hessian matrix of the energy with respect to  $S$  and  $Q$  can be obtained as

$$\begin{aligned} H_{s,Q}^\varepsilon &= \frac{64 r^{-(\Gamma+d-\theta)}}{s(2s-1)^{1/s} q^{2(s-1)}} \left\{ \frac{4(d-\theta)\Gamma}{r_0^{2(z-1-\theta/d)}} \right\}^{-\frac{2s-1}{s}} F \\ F &= z(z + d - \theta) r_+^z - \Gamma(2s-1)(z-2) q^{2s} r_+^{-(\Gamma+d-\theta)} \\ &= 4\pi z T + 2\Gamma[(z-1) + s(2-z)] q^{2s} r_+^{-(\Gamma+d-\theta)}. \end{aligned} \quad (50)$$

Assuming  $d > \theta$  and  $s > 1/2$ , one can easily see from Eq. (50) that the only factor in the Hessian matrix which can be negative is  $F$ .

First, we investigate the condition of having a stable black hole. In the case of linear Maxwell field ( $s = 1$ ),  $F > 0$  and therefore the linear charged Lifshitz black branes with hyperscaling violating factor is always stable in grand-canonical ensemble. In the case of nonlinear electromagnetic field ( $s > 1/2$ ),  $F$  is positive for  $1 < z \leq 2$  and therefore the black hole solution with  $1 < z \leq 2$  is stable in grand-canonical ensemble. Of course, one may remember that  $\Gamma$  should be positive and therefore  $d - \theta > (2s-1)(2-z)$ . However, for the case of  $z > 2$ ,  $F$  is positive provided:

$$s \leq \frac{z-1}{z-2},$$

or

$$z \leq \frac{2s-1}{s-1}.$$

It is worth mentioning that in this case since  $z$  is larger than two,  $s$  should be larger than 1.

Second, we investigate the unstable phase of nonlinear charged hyperscaling violating Lifshitz black branes. Indeed, the black brane solution is unstable provided

$$z > \frac{2s-1}{s-1},$$

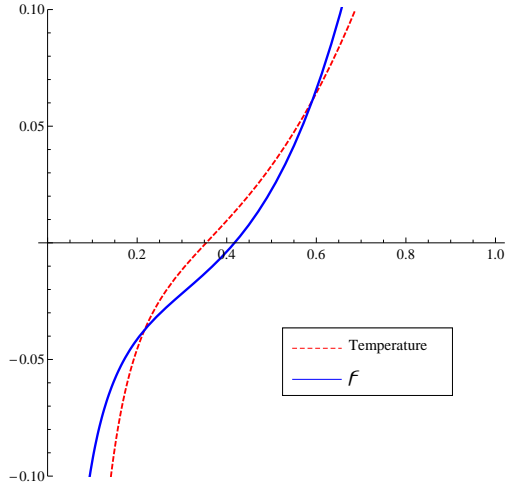


FIG. 1: Temperature (red,dotted) and  $F$  (blue,line) as functions of the horizon radius for  $q = 0.6$ ,  $d = 3$ ,  $z = 3$ ,  $s = 4$  and  $\theta = 2$ .

or

$$s > \frac{z-1}{z-2}.$$

Note that in either of the cases  $s$  is larger than 1. To be more clear, we plot a case which can have an unstable phase. As one can see in Fig 1, the small black branes  $r_{ext} \leq r_+ < r_{+max}$  is unstable, while the large black branes are stable. Note that in Fig. 1 the black branes with horizon radius in the range (0.354-0.418) are unstable.

## VII. CONCLUDING REMARKS

In this paper, we considered asymptotically Lifshitz black branes with hyperscaling violation in the presence of a massless nonlinear gauge field in dilaton gravity. Indeed, there are two massless electromagnetic fields coupled with the dilaton field. The first one which is linear accompanied with the dilaton field makes the desired asymptotic for the solution, while the second nonlinear gauge field gives charge to the solution. After presenting an analytic nonlinear charged hyperscaling violating Lifshitz black brane solution, we present the action in both the canonical and grand-canonical ensemble. Indeed, the appropriate action of a charged black hole in the grand-canonical and canonical ensembles are not the same. The appropriate actions in these ensembles in the absence of a dilaton field are given by Hawking and Ross for a linear electromagnetic field [16]. Its generalization to the case

of nonlinear gauge field in the absence of dilaton field has been introduced in [17]. Here, we generalized this action to the case of power-law electromagnetic theory in the presence of dilaton field and introduced the appropriate action for both the canonical and grand-canonical ensembles. Next, we generalized the counterterm method introduced in Ref. [15] to our case and used it to calculate the finite energy density. We also found the general thermodynamic relationship for the energy density in terms of the extensive thermodynamic quantities, entropy, and charge density, and their intensive conjugate quantities, temperature and electric potential. This result generalizes the well-known Smarr formula of AdS black holes and reduces to the Smarr formula of the 4-dimensional uncharged Lifshitz solution of Ref. [6]. Finally, we investigated the stability of the solutions in both the canonical and grand-canonical ensembles. Before, summarizing the results, it is worth mentioning the constraint on the parameters  $z$ ,  $\theta$ ,  $s$  and  $d$ . First, since the electric potential should vanish at infinity  $d - \theta + (2s - 1)(z - 2) > 0$ . Second, since  $f(r)$  should go to one as  $r$  goes to infinity,  $z + d - \theta > 0$ . Finally, since  $q_1$  should be real,  $z > 1$ . We found that for  $d > \theta$  while the solution is thermally stable in canonical ensemble, it can have an unstable phase in grand-canonical ensemble. Indeed, this instability is the effect of the nonlinearity of the electromagnetic field. We found that when the field is linear ( $s = 1$ ), the black brane solution is stable. We also found that for  $z \leq 2$ , the solution is stable even in grand-canonical ensemble.

The thermodynamics of rotating Lifshitz black branes in the presence of nonlinear massless gauge field has been investigated in [19]. In Ref. [19], the counterterm method has been generalized to the case of rotating Lifshitz solution. Here, we applied the counterterm method only to the case of static solutions. It would be interesting to generalize this method to the case of rotating black branes. Further work in this area will involve considering the thermodynamics of Lifshitz black branes of modified theory of gravity in the presence of a dilaton field with or without hyperscaling violating factor. These works are in progress.

## Acknowledgments

We thank Shiraz University Research Council. This work has been supported financially

by Research Institute for Astronomy and Astrophysics of Maragha, Iran.

---

- [1] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 505 (1998).
- [2] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Phys. Rev. Lett.* **101**, 031601 (2008); P. Kovtun and D. Nickel, *Phys. Rev. Lett.* **102**, 011602 (2009); C. P. Herzog, P. K. Kovtun, and D. T. Son, *Phys. Rev.* **79**, 066002 (2009); C. P. Herzog, *J. Phys. A* **42**, 343001 (2009); S. A. Hartnoll, *Class. Quant. Grav.* **26**, 224002 (2009).
- [3] K. Balasubramanian and J. McGreevy, *Phys. Rev. Lett.* **101**, 061601 (2008).
- [4] S. Kachru, X. Liu and M. Mulligan, *Phys. Rev. D* **78**, 106005 (2008); R. B. Mann, *J. High Energy Phys.* **06**, 075 (2009); K. Balasubramanian and J. McGreevy, *Phys. Rev. D* **80**, 104039 (2009); U. H. Danielsson, L. Thorlacius, *J. High Energy Phys.* **03**, 070 (2009).
- [5] M. H. Dehghani and R. B. Mann, *J. High Energy Phys.* **07**, 019 (2010); M. H. Dehghani and R. B. Mann, *Phys. Rev. D* **82**, 64019 (2010); W. G. Brenna, M. H. Dehghani, *Phys. Rev. D* **84**, 024012 (2011); R. B. Mann, M. H. Dehghani, R. B. Mann and R. Pourhasan, *Phys. Rev. D* **84**, 046002 (2011).
- [6] G. Bertoldi, B. A. Burrington and A. Peet, *Phys. Rev. D* **80**, 126003 (2009).
- [7] G. Bertoldi, B. A. Burrington and A. W. Peet, *Phys. Rev. D* **82**, 106013 (2010); G. Bertoldi, B. A. Burrington, A. W. Peet and I. G. Zadeh, *Phys. Rev. D* **83**, 126006 (2011).
- [8] S. S. Gubser and F. D. Rocha, *Phys. Rev. D* **81**, 046001 (2010); K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, *J. High Energy Phys.* **08**, 078 (2010); M. Cadoni, G. D'Appollonio and P. Pani, *J. High Energy Phys.* **03**, 100 (2010); M. Cadoni and P. Pani, *J. High Energy Phys.* **04**, 049 (2011); E. Perlmutter, *J. High Energy Phys.* **02**, 013 (2011); K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, *J. High Energy Phys.* **10**, 027 (2010); N. Iizuka, N. Kundu, P. Narayan and S. P. Trivedi, *arXiv:1105.1162 [hep-th]*; P. Berglund, J. Bhattacharyya and D. Mattingly, *J. High Energy Phys.* **08**, 042 (2012).
- [9] J. Tarrio, S. Vandoren, *J. High Energy Phys.* **09**, 017 (2011).
- [10] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, *J. High Energy Phys.* **11**, 151 (2010).
- [11] X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, *J. High Energy Phys.* **06**, 041

- (2012); M. Alishahiha and H. Yavartanoo, J. High Energy Phys. **11**, 034 (2012).
- [12] L. Huijse and S. Sachdev, Phys. Rev. D **84**, 026001 (2011); L. Huijse, S. Sachdev and B. Swingle, Phys. Rev. B. **85**, 035121 (2012).
- [13] M. Alishahiha, E. . Colgin, H. Yavartanoo J. High Energy Phys. **11**, 137 (2012); M. Alishahiha, A. Faraji Astaneh and M. R. Mohammadi Mozaffara, arXiv:1401.2807.
- [14] M. Hassaine and C. Martinez, Class. Quantum Grav. **25**, 195023 (2008).
- [15] S. F. Ross and O. Saremi, J. High Energy Phys. **09**, 009 (2009).
- [16] S. W. Hawking and S. F. Ross, Phys. Rev. D **52**, 5865 (1995).
- [17] M. H. Dehghani, Ch. Shakuri and M. H. Vahidinia, Phys. Rev. D **87**, 084013 (2013).
- [18] M. Cvetič, S.S. Gubser, J. High Energy Phys. **04**, 024 (1999); M. M. Caldarelli, G. Cognola, D. Klemm, Class. Quantum Grav. **17**, 399 (2000); M. H. Dehghani and A. Khodam-Mohammadi, Phys. Rev. D **67**, 084006 (2003).
- [19] M. H. Dehghani, and Sh. Asnafi, Phys. Rev. D **84**, 064038 (2011).